EM 20

"Y and E"
KRAFT" FOLDER

Better folders for better files
K 9474 2/5
Send your Order to the nearest "Y and E"
Representatives or to our Home Office
YAWMAN AND ERBE MFG. CO.
Main Factories and Executive Offices
ROCHESTER, N. Y.
Branches and Agents in all Principal Cities

New York University
Washington Square College
Mathematics Research Group
Research Report No. EM-20
under
Contract No.AF-19(122)-42

DETERMINATION OF REFLECTED AND TRANSMITTED FIELDS BY GEOMETRICAL OPTICS II

by
Joseph B. Keller
and
Stanley Preiser

Written by:

Joseph B. Keller
Joseph B. Keller

Stanley Preiser
Stanley Preiser

Morris Kline
Morris Kline
Project Director

Title page
10 numbered pages
April, 1950

CONTENTS

Article	Page
Abstract	i
1. Introduction	1
2. Formulation of the Problem	2
3. Calculation of Jacobian	4
4. Discussion for a Point Source	8
5. Discussion for a Normally Incident Wavefront	9

ABSTRACT

An extension of ordinary geometrical optics (or acoustics), makes it possible to evaluate the field in tensity, phase and polarization. By this method we have considered an incident field having an arbitrary wavefront which impinges on an arbitrary interface between two media, and have computed the amplitudes of the reflected and transmitted fields. The result includes the special cases of plane and spherical incident wavefronts which were treated previously.

1. Introduction

If in geometrical optics we assume that the principle of the conservation of energy applies to an infinitesimal tube of rays we conclude that the field components along the ray vary inversely as the square root of the normal cross-sectional area of this tube. Since the rays are determined by Fermat's principle these rays can be found and then the energy assumption enables us to determine the field along the ray if it is known at one point on this ray. Luneberg proved that solutions of Maxwell's equations for small wave lengths behave in accordance with the above assumption.* A similar result has been proved in acoustics.**

We apply the above procedure to the problem of determining the field due to an incident field having an arbitrary wave front which impinges on an interface separating two homogeneous, isotropic media. It then becomes necessary to calculate the ratio of the infinitesimal areas at the two ends of a tube of rays where the rays of the tube are reflected from or transmitted by the interface. This area ratio is precisely the Jacobian of the mapping of one area onto the other.

In I*** it is shown that the Jacobian required in this problem is the same as that of the mapping of the tangent plane to the reflecting surface onto a parallel plane through the point of observation. The mapping is that given by the incident rays which intersect the tangent plane and are then reflected or refracted at the interface, finally intersecting the plane through the observation point. The incident rays are the lines normal to the incident wave front. In I the incident wave front was assumed to be a point, corresponding to a point source. In this paper we consider an arbitrary incident wave front and compute the Jacobian described above.

In the next section (2) the geometry and the mapping are described; in section 3 the Jacobian is computed. In section 4 it is shown that this result includes that of I when the incident surface is specialized to a point. In section 5 the Jacobian is specialized to the case of an arbitrary wave front incident normally on the reflecting surface.

---

* Luneberg: Lecture Notes in Electromagnetic Theory, New York University.

** Keller, J. B.: Lecture Notes in Fluid Dynamics, 1949-50, New York University.

*** Journal of the Optical Society of America - J.B. Keller and H. B. Keller, Vol. 40 No. 1, January, 1950. (Hereafter(I) will refer to this article.) This material was included in Research Report No. EM-13, N.Y.U. Mathematics Research Group.

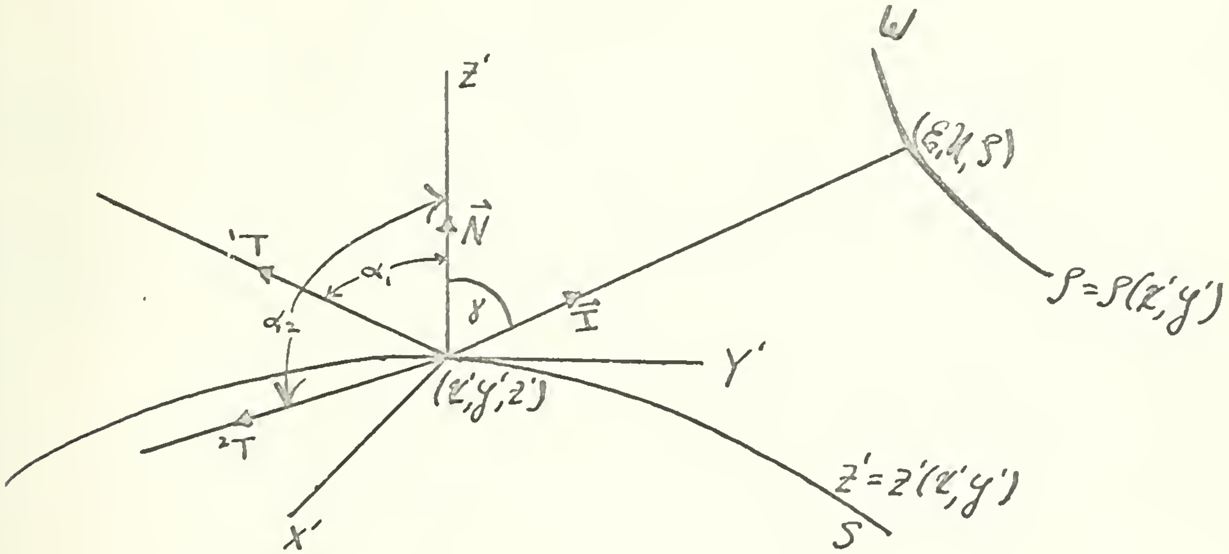
2. Formulation of the Problem

Let $\vec{N}(x',y',z')$ be a unit vector normal to $z' = z'(x',y')$ and pointing into the first medium. $\vec{I}(x',y',z')$ is the unit vector pointing from (x',y',z') to a point (ξ,η,ζ) on the incident wavefront. The direction of the reflected ray at (x',y',z') is given bythe unit vector ${}^{1}T(x',y',z')$ and that of the transmitted ray by ${}^{2}T(x',y',z')$ where, from the laws of reflection and refraction (See I):

$$ {}^{\lambda}\vec{T}(x',y',z') = -n^{(\lambda-1)}\vec{I} + \left[n^{(\lambda-1)}(\vec{I}\cdot\vec{N}) - (-1)^{\lambda}\left\{1 - n^{2(\lambda-1)}\left[1-(\vec{I}\cdot\vec{N})^{2}\right]^{1/2}\right\}\right]\vec{N} $$

(For the sake of compactness let a_{λ} be the coefficient of the vector $\vec{I}$ and b_{λ} the coefficient of the vector $\vec{N}$ or: ${}^{\lambda}T(x',y',z') = a_{\lambda}\vec{I} + b_{\lambda}\vec{N}$)

Here $n = \frac{\sin\alpha_{\lambda}}{\sin\gamma}$; α_1 = the angle of reflection, α_2 = the angle of refraction, γ = the angle of incidence, and n = the index of refraction. The angles and unit vectors are shown below:



The unit vector $\vec{I}$ along the incident ray and the unit normal $\vec{N}$ to the reflecting surface have the components:

$$ \vec{I}(x',y',z') = \left(\frac{\xi - x'}{\rho(x',y',z')}, \frac{\eta - y'}{\rho(x',y',z')}, \frac{\zeta - z'}{\rho(x',y',z')}\right) $$

$$ \vec{N}(x',y',z') = \frac{-\frac{dz'}{dx'}}{\eta(x',y',z')}, \frac{-\frac{dz'}{dy'}}{\eta(x',y',z')}, \frac{1}{\eta(x',y',z')} $$

where: $\rho^2(x',y',z') = (\xi-x')^2 + (\eta-y')^2 + (\zeta-z')^2$

and $n^2(x',y',z') = 1 + \left(\frac{\partial z'}{\partial x'}\right)^2 + \left(\frac{\partial z'}{\partial y'}\right)^2$

The coordinate system must be so chosen that $\vec{N}$ points into the first medium.

The equation of the reflected or transmitted ray through a point on the reflecting surface is given by:

$$\frac{x-x'}{{}^{r}T_x} = \frac{y-y'}{{}^{r}T_y} = \frac{z-z'}{{}^{r}T_z}$$

or rewriting we get :

$$x = x' + (z-z')\frac{{}^{r}T_x(x',y',z')}{{}^{r}T_z(x',y',z')}$$

(1)

$$y = y' + (z-z')\frac{{}^{r}T_y(x',y',z')}{{}^{r}T_z(x',y',z')}$$

Here the (x,y,z) designate the variable point on the reflected or transmitted ray and ${}^{r}T_x, {}^{r}T_y, {}^{r}T_z$ are the direction numbers of this line.

The coordinate system has been chosen with the origin on the reflecting surface and the z' axis along the normal. Thus $z'=0$ represents the tangent plane to the reflecting surface at the origin and $z = \text{Constant}$ represents a parallel plane.

Hence the equation (1) with $z'=0$ and $z=\text{Const.}$ yield the mapping of $x'y'$ on the tangent plane to xy on a parallel plane by means of reflected or transmitted rays. As mentioned in the introduction and shown I the Jacobian J of the inverse mapping, evaluated at the origin, determines the area ratio necessary to compute the amplitude of the reflected or transmitted field. This Jacobian is given by:

(2) $$J^{-1} = \left\{\frac{\partial x}{\partial x'}\frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'}\frac{\partial y}{\partial x'}\right\}$$

or

(3) $$J^{-1} = 1 + z\left[\frac{\partial}{\partial x'}\left(\frac{{}^{r}T_x}{{}^{r}T_z}\right) + \frac{\partial}{\partial y'}\left(\frac{{}^{r}T_y}{{}^{r}T_z}\right)\right]$$

$$+ z^2\left[\frac{\partial}{\partial x'}\left(\frac{{}^{r}T_x}{{}^{r}T_z}\right)\frac{\partial}{\partial y'}\left(\frac{{}^{r}T_y}{{}^{r}T_z}\right) - \frac{\partial}{\partial y'}\left(\frac{{}^{r}T_x}{{}^{r}T_z}\right)\frac{\partial}{\partial x'}\left(\frac{{}^{r}T_y}{{}^{r}T_z}\right)\right]$$

Equation (3) is obtained from Eq. (2) by performing the indicated differentiations in Eq. (1) and substitution into Eq. (2) remembering that these partial derivatives are to be evaluated at the origin.

3. Calculation of the Jacobian

In order to compute the Jacobian presented in Eq. (3) we must first evaluate the components of ${}^{\alpha}\overline{T}$ and its partial derivatives at the origin. These become:

$${}^{\alpha}T_x = \frac{a_\alpha \xi}{D} \;;\quad {}^{\alpha}T_y = \frac{a_\alpha \eta}{D} \;;\quad {}^{\alpha}T_z = \frac{a_\alpha \zeta}{D} + b_\alpha ; \text{ where } D = \sqrt{\xi^2+\eta^2+\zeta^2}$$

$$\frac{\partial({}^{\alpha}T_x)}{\partial x'} = \frac{a_\alpha\left(\frac{\partial\xi}{\partial x'}-1\right)D^2 - a_\alpha\xi\left[\xi\left(\frac{\partial\xi}{\partial x'}-1\right)+\eta\frac{\partial\eta}{\partial x'}+\zeta\frac{\partial\zeta}{\partial x'}\right]}{D^3} - b_\alpha\frac{\partial^2 z'}{\partial x'^2}$$

$$\frac{\partial({}^{\alpha}T_y)}{\partial x'} = \frac{a_\alpha\frac{\partial\eta}{\partial x'}D^2 - a_\alpha\eta\left[\xi\left(\frac{\partial\xi}{\partial x'}-1\right)+\eta\frac{\partial\eta}{\partial x'}+\zeta\frac{\partial\zeta}{\partial x'}\right]}{D^3} - b_\alpha\frac{\partial^2 z'}{\partial y'\partial x'}$$

$$\frac{\partial({}^{\alpha}T_x)}{\partial y'} = \frac{a_\alpha\frac{\partial\xi}{\partial y'}D^2 - a_\alpha\xi\left[\xi\frac{\partial\xi}{\partial y'}+\eta\left(\frac{\partial\eta}{\partial y'}-1\right)+\zeta\frac{\partial\zeta}{\partial y'}\right]}{D^3} - b_\alpha\frac{\partial^2 z'}{\partial y'\partial x'}$$

$$\frac{\partial({}^{\alpha}T_y)}{\partial y'} = \frac{a_\alpha\left(\frac{\partial\eta}{\partial y'}-1\right) - a_\alpha\eta\left[\xi\frac{\partial\xi}{\partial y'}+\eta\left(\frac{\partial\eta}{\partial y'}-1\right)+\zeta\frac{\partial\zeta}{\partial y'}\right]}{D^3} - b_\alpha\frac{\partial^2 z}{\partial y'^2}$$

$$\frac{\partial}{\partial x'}({}^{\alpha}T_z) = \frac{a_\alpha\frac{\partial\zeta}{\partial x'}D^2 - a_\alpha\zeta\left[\xi\left(\frac{\partial\xi}{\partial x'}-1\right)+\eta\frac{\partial\eta}{\partial x'}+\zeta\frac{\partial\zeta}{\partial x'}\right]}{D^3}$$

$$+\left(\frac{a_\alpha^2\zeta}{D\cos\alpha_\alpha}-a_\alpha\right)\left[\frac{\left(-\xi\frac{\partial^2 z'}{\partial x'^2}-\eta\frac{\partial^2 z'}{\partial y'\partial x'}+\frac{\partial\zeta}{\partial x'}\right)D^2}{D^3}\right]$$

$$+\left(\frac{a_\alpha^2\zeta}{D\cos\alpha_\alpha}-a_\alpha\right)\left[\frac{-\zeta\left(\xi\left\{\frac{\partial\xi}{\partial x'}-1\right\}+\eta\frac{\partial\eta}{\partial x'}+\zeta\frac{\partial\zeta}{\partial x'}\right)}{D^3}\right]$$

$$\frac{\partial({}^{\alpha}T_z)}{\partial y'} = \frac{a_\alpha\frac{\partial\zeta}{\partial y'}D^2 - a_\alpha\zeta\left[\xi\frac{\partial\xi}{\partial y'}+\eta\left(\frac{\partial\eta}{\partial y'}-1\right)+\zeta\frac{\partial\zeta}{\partial y'}\right]}{D^3} + \left(\frac{a_\alpha^2\zeta}{D\cos\alpha_\alpha}-a_\alpha\right)\frac{\left[-\xi\frac{\partial^2 z}{\partial x'\partial y'}-\eta\frac{\partial^2 z}{\partial y'^2}+\frac{\partial\zeta}{\partial y'}\right]D^2}{D^3}$$

$$+\left(\frac{a_\alpha^2\zeta}{D\cos\alpha_\alpha}-a_\alpha\right)\left[\frac{-\zeta\left(\xi\frac{\partial\xi}{\partial y'}+\eta\left(\frac{\partial\eta}{\partial y'}-1\right)+\zeta\frac{\partial\zeta}{\partial y'}\right)}{D^3}\right]$$

These expressions involve the partial derivatives $\left(\frac{\partial \xi}{\partial x'}, \frac{\partial \xi}{\partial y'}, \frac{\partial \eta}{\partial x'}, \frac{\partial \eta}{\partial y'}, \frac{\partial \zeta}{\partial x'}, \frac{\partial \zeta}{\partial y'}\right)$ To solve for these six partial derivatives we must first write the equation of the line normal to the wave front. This line is given by:

$$\frac{x'-\xi}{\frac{\partial \zeta}{\partial \xi}} = \frac{y'-\eta}{\frac{\partial \zeta}{\partial \eta}} = \frac{z'-\zeta}{-1} \quad or \quad \xi = x' + \frac{\partial \zeta}{\partial \xi}(z'-\zeta)$$

$$\eta = y' + \frac{\partial \zeta}{\partial \eta}(z'-\zeta)$$

We first differentiate with respect to x', y' and solve the resulting equations for the partial derivatives $\left(\frac{\partial \xi}{\partial x'}, \frac{\partial \xi}{\partial y'}, \frac{\partial \eta}{\partial x'}, \frac{\partial \eta}{\partial y'}\right)$ The resulting expressions evaluated at the origin $x'=y'=z'=0$ are:

$$\frac{\partial \xi}{\partial x'} = \frac{1+\left(\frac{\partial \zeta}{\partial \eta}\right)^2 + \zeta\left(\frac{\partial^2 \zeta}{\partial \eta^2}\right)}{\left(1+\left(\frac{\partial \zeta}{\partial \eta}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \eta^2}\right)\right)\left(1+\left(\frac{\partial \zeta}{\partial \xi}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \xi^2}\right)\right)-\left(\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}+\zeta\frac{\partial^2 \zeta}{\partial \xi \partial \eta}\right)^2}$$

$$\frac{\partial \eta}{\partial x'} = \frac{\partial \xi}{\partial y'} = -\frac{\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}+\zeta\frac{\partial^2 \zeta}{\partial \eta \partial \xi}}{\left[1+\left(\frac{\partial \zeta}{\partial \eta}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \eta^2}\right)\right]\left[1+\left(\frac{\partial \zeta}{\partial \xi}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \xi^2}\right)\right]-\left[\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}+\zeta\frac{\partial^2 \zeta}{\partial \xi \partial \eta}\right]^2}$$

$$\frac{\partial \eta}{\partial y'} = \frac{1+\left(\frac{\partial \zeta}{\partial \xi}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \xi^2}\right)}{\left[1+\left(\frac{\partial \zeta}{\partial \eta}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \eta^2}\right)\right]\left[1+\left(\frac{\partial \zeta}{\partial \xi}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \xi^2}\right)\right]-\left[\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}+\zeta\frac{\partial^2 \zeta}{\partial \xi \partial \eta}\right]^2}$$

The remaining partial derivatives $\frac{\partial \zeta}{\partial x'}, \frac{\partial \zeta}{\partial y'}$ are given by the chain rule for differentiation and at the origin become:

$$\frac{\partial \zeta}{\partial x'} = \frac{\frac{\partial \zeta}{\partial \xi}\left(1+\left(\frac{\partial \zeta}{\partial \eta}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \eta^2}\right)\right)-\frac{\partial \zeta}{\partial \eta}\left[\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}+\zeta\frac{\partial^2 \zeta}{\partial \eta \partial \xi}\right]}{\left[1+\left(\frac{\partial \zeta}{\partial \eta}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \eta^2}\right)\right]\left[1+\left(\frac{\partial \zeta}{\partial \xi}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \xi^2}\right)\right]-\left[\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}+\zeta\frac{\partial^2 \zeta}{\partial \xi \partial \eta}\right]^2}$$

$$\frac{\partial \zeta}{\partial y'} = \frac{\frac{\partial \zeta}{\partial \eta}\left[1+\left(\frac{\partial \zeta}{\partial \xi}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \xi^2}\right)\right]-\frac{\partial \zeta}{\partial \xi}\left[\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}+\zeta\frac{\partial^2 \zeta}{\partial \eta \partial \xi}\right]}{\left[1+\left(\frac{\partial \zeta}{\partial \eta}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \eta^2}\right)\right]\left[1+\left(\frac{\partial \zeta}{\partial \xi}\right)^2+\zeta\left(\frac{\partial^2 \zeta}{\partial \xi^2}\right)\right]-\left[\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}+\zeta\frac{\partial^2 \zeta}{\partial \xi \partial \eta}\right]^2}$$

The expressions obtained above and those to follow can be greatly simplified and more easily interpreted by introducing geometrical quantities. We therefore introduce the coefficients of the first and second differential forms of the incident wave front and the reflecting surface. These coefficients are given by:****

$$e = 1+\left(\frac{\partial\rho}{\partial\xi}\right)^2 \qquad E = 1+\left(\frac{\partial z'}{\partial x'}\right)^2$$

$$f = \frac{\partial\rho}{\partial\xi}\frac{\partial\rho}{\partial\eta} \qquad F = \frac{\partial z'}{\partial x'}\frac{\partial z'}{\partial y'}$$

$$g = 1+\left(\frac{\partial\rho}{\partial\eta}\right)^2 \qquad G = 1+\left(\frac{\partial z'}{\partial y'}\right)^2$$

$$l = \frac{1}{\theta}\frac{\partial^2\rho}{\partial\xi^2} \qquad L = \frac{\partial^2 z'}{\partial x'^2}$$

$$m = \frac{1}{\theta}\frac{\partial^2\rho}{\partial\xi\partial\eta} \qquad M = \frac{\partial^2 z'}{\partial x'\partial y'}$$

$$n = \frac{1}{\theta}\frac{\partial^2\rho}{\partial\eta^2} \qquad N = \frac{\partial^2 z'}{\partial y'^2}$$

$$\theta = \left\{1+\left(\frac{\partial\rho}{\partial\xi}\right)^2+\left(\frac{\partial\rho}{\partial\eta}\right)^2\right\}^{1/2}$$

$$h = \frac{1}{2}\frac{en+gl-2fm}{eg-f^2} \qquad H = \frac{1}{2}\frac{EN+GL-2FM}{EG-F^2} \text{(mean Curvature)}$$

$$k = \frac{ln-m^2}{eg-f^2} \qquad K = \frac{LN-M^2}{EG-F^2} \text{(Gaussian Curvature)}$$

$$\frac{\partial\xi}{\partial x'} = \frac{(g+\rho n\theta)}{\left[\qquad\right]} \;;\; \frac{\partial\xi}{\partial y'} = \frac{\partial\eta}{\partial x'} = -\frac{(f+\rho m\theta)}{\left[\qquad\right]} \;;\; \frac{\partial\eta}{\partial y'} = \frac{(e+\rho l\theta)}{\left[\qquad\right]}$$

$$\frac{\partial\rho}{\partial x'} = \frac{\frac{\partial\rho}{\partial\xi}(g+\rho n\theta)-\frac{\partial\rho}{\partial\eta}(f+\rho m\theta)}{\left[\qquad\right]} \;;\; \frac{\partial\rho}{\partial y'} = \frac{\frac{\partial\rho}{\partial\eta}(e+\rho l\theta)-\frac{\partial\rho}{\partial\xi}(f+\rho m\theta)}{\left[\qquad\right]}$$

$$\left[\qquad\right] = (eg-f^2)+\rho\theta(en+gl-2fm)+\rho^2\theta^2(ln-m^2)$$

This bracketed expression is equal to $eg-f^2$ multiplied by the expansion ratio of the incident wave front. This expansion ratio is equal to an area on the incident wave front divided by the corresponding area after the wave front reaches the reflecting surface.**

Upon substituting these quantities into the Jacobian and after many algebraic manipulations we arrive at the final expression for it:

**** J.J. Stoker, Lecture Notes on Differential Geometry, N.Y.U., 1942-43.

$$J = I +$$

$$\frac{Z}{\cos\alpha_1}\Bigg[\frac{a_1}{D^3}\Big\{\xi^2+\eta^2-2D^2+\frac{(g+\rho n\theta)}{[\quad]}\Big(D^2-\xi^2-\xi\rho\frac{d\rho}{d\xi}\Big)+\frac{(e+\rho l\theta)}{[\quad]}\Big(D^2-\eta^2-\eta\rho\frac{d\rho}{d\eta}\Big)+\frac{(f+\rho m\theta)}{[\quad]}\Big(\xi\rho\frac{d\rho}{d\eta}+2\xi\eta+\eta\rho\frac{d\rho}{d\xi}\Big)\Big\}-b_1\Big\{L+N+\frac{a_1^2(\xi^2L+2\xi\eta M+\eta^2N)}{D^2\cos^2\alpha_1}\Big\}$$

$$-\frac{a_1^3\xi\rho}{D^5\cos^2\alpha_1}\Big\{\frac{(g+\rho n\theta)}{[\quad]}\Big(\frac{d\rho}{d\xi}[D^2-\rho^2]-\xi\rho\Big)-\frac{(f+\rho m\theta)}{[\quad]}\Big(\frac{d\rho}{d\eta}[D^2-\rho^2]-\rho\eta\Big)+\rho\xi\Big\}-\frac{a_1^3\eta\rho}{D^5\cos^2\alpha_1}\Big\{\frac{(e+\rho l\theta)}{[\quad]}\Big(\frac{d\rho}{d\eta}[D^2-\rho^2]-\rho\eta\Big)-\frac{(f+\rho m\theta)}{[\quad]}\Big(\frac{d\rho}{d\xi}[D^2-\rho^2]-\rho\xi\Big)+\rho\eta\Big\}\Bigg]$$

$$\frac{1}{D^6}\Bigg[a_1^2D^4\Big\{\frac{1-(e+\rho l\theta)-(g+\rho n\theta)}{[\quad]}+1\Big\}+b_1^2D^6(LN-M^2)+a_1b_1D^5\Big\{L+N-\frac{L(e+\rho l\theta)+2M(f+\rho m\theta)+N(g+\rho n\theta)}{[\quad]}\Big\}$$

$$+a_1b_1D^3\Big\{\frac{(e+\rho l\theta)}{[\quad]}\Big(\eta+\rho\frac{d\rho}{d\eta}\Big)(\eta L-\xi M)+\frac{(g+\rho n\theta)}{[\quad]}\Big(\xi+\rho\frac{d\rho}{d\xi}\Big)(\xi N-\eta M)-\frac{(f+\rho m\theta)}{[\quad]}\Big[\Big(\xi+\rho\frac{d\rho}{d\xi}\Big)(\eta L-\xi M)+\Big(\eta+\rho\frac{d\rho}{d\eta}\Big)(\xi N-\eta M)\Big]+\eta(\xi M-\eta L)+\xi(\eta M-\xi N)\Big\}$$

$$+a_1^2D^2\Big\{\frac{(e+\rho l\theta)}{[\quad]}\Big(\eta^2+\xi^2+\eta\rho\frac{d\rho}{d\eta}\Big)+\frac{(g+\rho n\theta)}{[\quad]}\Big(\eta^2+\xi^2+\rho\xi\frac{d\rho}{d\xi}\Big)-\frac{(f+\rho m\theta)}{[\quad]}\Big(\eta\rho\frac{d\rho}{d\xi}+\xi\rho\frac{d\rho}{d\eta}\Big)-\frac{\Big(\xi^2+\eta^2+\xi\rho\frac{d\rho}{d\xi}+\eta\rho\frac{d\rho}{d\eta}\Big)}{[\quad]}-(\xi^2+\eta^2)\Big\}\Bigg]$$

$$\frac{Z^2}{\cos^2\alpha_1}\Bigg[a_1^2D^4\Big\{\frac{\xi\frac{d\rho}{d\xi}(1-[g+\rho n\theta])+\eta\frac{d\rho}{d\eta}(1-[e+\rho l\theta])+(f+\rho m\theta)\Big(\xi\frac{d\rho}{d\eta}+\eta\frac{d\rho}{d\xi}\Big)}{[\quad]}\Big\}+\frac{a_1b_1^2D^3}{\cos\alpha_1}\Big\{D^2(\xi^2+\eta^2)(M^2-LN)+\rho(\xi^2N-2\xi\eta M+\eta^2L)\Big\}$$

$$+\frac{a_1b_1^2D^3}{\cos\alpha_1}\Big\{\frac{(g+\rho n\theta)}{[\quad]}(\xi N-\eta M)\Big[(\eta^2+\xi^2)\frac{d\rho}{d\xi}-\xi\rho\Big]+\frac{(e+\rho l\theta)}{[\quad]}(\eta L-\xi M)\Big[(\eta^2+\xi^2)\frac{d\rho}{d\eta}-\rho\eta\Big]+\frac{(f+\rho m\theta)}{[\quad]}\Big[(\eta M-\xi N)(\eta^2+\xi^2)\frac{d\rho}{d\eta}+(\xi M-\eta L)(\eta^2+\xi^2)\frac{d\rho}{d\xi}+\rho\xi\eta(L+N)-\rho M(\xi^2+\eta^2)\Big]\Big\}$$

$$-\frac{a_1}{D^2\cos\alpha_1}\Bigg|-\frac{a_1^2b_1D^2}{\cos\alpha_1}\Bigg[\frac{(\eta^2+\xi^2)}{[\quad]}\Big(\xi\frac{d\rho}{d\xi}-\rho+\eta\frac{d\rho}{d\eta}\Big)-\rho(\xi^2+\eta^2)+\frac{(e+\rho l\theta)}{[\quad]}\Big[\eta^2\rho-D^2\xi(L\xi+\eta M)-(\eta^2+\xi^2)\eta\frac{d\rho}{d\eta}\Big]+\frac{(g+\rho n\theta)}{[\quad]}\Big[\xi^2\rho-D^2\eta(\xi M+\eta N)-(\eta^2+\xi^2)\xi\frac{d\rho}{d\xi}\Big]+$$

$$+\frac{(f+\rho m\theta)}{[\quad]}\Big[(\eta^2+\xi^2)\Big(\eta\frac{d\rho}{d\xi}+\xi\frac{d\rho}{d\eta}\Big)-D^2(\eta[\xi L+\eta M]+\xi[\eta N+\xi M])\Big]+D^2[\xi(\xi L+\eta M)+\eta(\xi M+\eta N)]\Bigg]$$

$$-a_1^2D^2\Big\{\frac{\rho(\eta^2+\xi^2)+\rho^2\Big(\eta\frac{d\rho}{d\eta}+\xi\frac{d\rho}{d\xi}\Big)}{[\quad]}+\rho(\xi^2+\eta^2)+\rho^2\frac{(f+\rho m\theta)}{[\quad]}\Big(\xi\frac{d\rho}{d\eta}+\eta\frac{d\rho}{d\xi}\Big)-\rho\frac{(g+\rho n\theta)}{[\quad]}\Big(\eta^2+\xi^2+\xi\rho\frac{d\rho}{d\xi}\Big)-\rho\frac{(e+\rho l\theta)}{[\quad]}\Big(\eta^2+\xi^2+\eta\rho\frac{d\rho}{d\eta}\Big)\Big\}+$$

$$+a_1b_1D^3\Big\{\frac{(g+\rho n\theta)}{[\quad]}\Big[\xi\rho(\xi N-\eta M)+\rho^2\frac{d\rho}{d\xi}(\xi N-\eta M)\Big]+\frac{(e+\rho l\theta)}{[\quad]}(\eta L-\xi M)\Big(\eta\rho+\rho^2\frac{d\rho}{d\eta}\Big)-\rho(\xi^2N-2\xi\eta M+\eta^2L)$$

$$\frac{(f+\rho m\theta)}{[\quad]}\Big[\rho^2\frac{d\rho}{d\xi}(\xi M-\eta L)+\rho^2\frac{d\rho}{d\eta}(\eta M-\xi N)-\xi\eta\rho(L+N)+\rho(\xi^2+\eta^2)M\Big]\Big\}$$

4. Discussion for the Point Source

We will consider the special form which the Jacobian assumes when the incident wave front is a point. This is the case if the incident field is produced by a point source and this case was treated in I. We wish to show that our result becomes identical with the result of I. To this end we characterize a point wave front by stating that its curvatures are infinite and thus $\left(\frac{\partial^2 S}{\partial\xi^2}, \frac{\partial^2 S}{\partial\xi\partial\eta}, \frac{\partial^2 S}{\partial\eta^2}\right)$ are infinite. Taking account of this Eq.(4) becomes:

$$J^{-1} = 1 +$$

$$\frac{z}{\cos\alpha_1}\left\{\frac{a_1}{D^3}(\xi^2+\eta^2-2D^2) - \frac{a_1^3\rho^2(\xi^2+\eta^2)}{D^5\cos^2\alpha_1} - b_1\left(L+N+\frac{a_1^2[\xi^2L+2\xi\eta M+\eta^2N]}{D^2\cos^2\alpha_1}\right)\right\} +$$

$$\frac{z^2}{\cos^2\alpha_1}\left\{\begin{aligned}&\frac{a_1^2}{D^2} + b_1(LN-M^2) + \frac{a_1b_1}{D}(L+N) + \frac{a_1b_1}{D^3}(2\xi\eta M-\eta^2L-\xi^2N) - \frac{a_1^2(\xi^2+\eta^2)}{D^4}\\ &-\frac{a_1^2b_1^2D^3}{D^7\cos^2\alpha_1}[D^2(\xi^2+\eta^2)(M^2-LN)+\rho(\xi^2N-2\xi\eta M+\eta^2L)] + \frac{a_1^3(\xi^2+\eta^2)\rho}{D^5\cos\alpha_1}\\ &+\frac{a_1^3b_1}{D^5\cos^2\alpha_1}[D^2(\xi^2L+2\eta\xi M+\eta^2N)-\rho(\xi^2+\eta^2)] + \frac{a_1^2b_1}{D^4\cos\alpha_1}[\xi^2N-2\xi\eta M+\eta^2L]\end{aligned}\right\}$$

Eq.(5)

$$J^{-1} = 1 + \frac{z}{\cos\alpha_1}\left\{-\frac{a_1}{D}\left[1+\frac{\rho^2}{D^2}+\frac{a_1^2(\xi^2+\eta^2)\rho^2}{D^4\cos^2\alpha_1}\right] - b_1(L+N) - \frac{a_1^2b_1(\xi^2L+2\xi\eta M+\eta^2N)}{D^2\cos^2\alpha_1}\right\}$$

$$+\frac{z^2}{\cos^2\alpha_1}\left\{\begin{aligned}&\frac{a_1^2}{D^2}\left[\frac{\rho^2}{D^2}+\frac{a_1^2(\xi^2+\eta^2)\rho^2}{D^4\cos^2\alpha_1}\right] + \frac{a_1b_1(L+N)}{D} + b_1^2(LN-M^2)\left[1+\frac{a_1^2(\xi^2+\eta^2)}{D^2\cos^2\alpha_1}\right]\\ &+\frac{a_1^3b_1}{D^3\cos^2\alpha_1}(\xi^2L+2\eta\xi M+\eta^2N) - \frac{a_1b_1}{D^3}\left(1-\frac{a_1^2\rho^2}{D^2\cos^2\alpha_1}\right)(\xi^2N-2\xi\eta M+\eta^2L)\end{aligned}\right\}$$

This is identical with the result in I and its consequences were discussed there.

5. Discussion for a Normally Incident Wave Front

Another special case of importance is that in which the incident ray is normally incident on the reflecting surface. In this case we have $e = g = \Theta = 1$ and $\xi = \eta = f = 0$. With these values Eq.(4) becomes:

$$J^{-1} = 1 + \frac{z}{\cos\alpha_1}\left\{\frac{a_1}{\rho^3}\left[\frac{\rho^2(1+\rho n)}{[\quad]} + \frac{\rho^2(1+\rho l)}{[\quad]} - 2\rho^2\right] - b_1(L+N)\right\}$$

$$+\frac{z^2}{\cos^2\alpha_1}\left\{\begin{array}{l}\frac{a_1^2}{\rho^2}\left[\frac{1-(1+\rho l)-(1+\rho n)}{[\quad]}+1\right] + b_1^2(LN-M^2) \\ -\frac{a_1 b_1}{\rho}\left[\frac{L(1+\rho l)+2Mm\rho+N(1+\rho n)}{[\quad]} - (L+N)\right]\end{array}\right\}$$

or

$$J^{-1} = 1 + |z|\left\{-a_1\frac{[(n+l)+2\rho(nl-m^2)]}{1+\rho(n+l)+\rho^2(nl-m^2)} - b_1(L+N)\right\}$$

Eq. (6)

$$+ z^2\left\{\begin{array}{l}\frac{a_1^2(nl-m^2)}{1+\rho(n+l)+\rho^2(nl-m^2)} + b_1(LN-M^2) \\ + a_1 b_1\frac{[(2Mm-Ln-Nl)-\rho(nl-m^2)(L+N)]}{1+\rho(n+l)+\rho^2(nl-m^2)}\end{array}\right\}$$

where $\cos\alpha_1 = \begin{cases} 1 & \text{for reflection} \\ -1 & \text{for transmission}\end{cases}$ and z has the same sign as $\cos\alpha_1$.

In addition $[\quad] = 1 + \rho(n+l) + \rho^2(nl-m^2)$

Finally, inserting the expressions for the curvatures we get:

$$J^{-1} = 1 + |z|\left\{\frac{-2a_1(h+\rho k)}{1+2\rho h+\rho^2 k^2} - 2b_1 H\right\}$$

$$+ z^2\left\{\frac{a_1^2 k}{1+2\rho h+\rho^2 k^2} + b_1^2 K + a_1 b_1[(2Mm-Ln-Nl)-2\rho Hk]\right\}$$

In this special case the coefficients a_λ and b_λ have the following values. $a_1 = -1$; $a_2 = -\underline{n}$; $b_1 = 2$; $b_2 = +\underline{n} - 1$

The main result of this paper Eq. (4) and its special cases Eq. (5) (point source) and Eq. (6) (normal incidence), seem to depend on the relevant geometrical quantities in the expected way. Unfortunately, however, the expressions are so complicated that it seems most reasonable to apply them to special cases of interest rather than to attempt a general study of their properties.

References

* Luneberg, R. K. - Lecture Notes in Electromagnetic Theory, New York University.

** Keller, J.B. - Lecture Notes in Fluid Dynamics, 1949-50.

*** Journal of the Optical Society of America, Vol. 40, Number 1, Jan. 1950 Keller, J.B. and H. B. Keller, Determination of Reflected and Transmitted Fields By Geometrical Optics. This material was included in Research Report No. EM-13, N.Y.U. Mathematics Research Group.

**** Stoker, J. J. - Lecture Notes in Differential Geometry, New York University, 1942-43.

DATE DUE

JAN 21 '64

NOV 10 '64

JAN 26

ILL - 2·27·87

GAYLORD PRINTED IN U.S.A.

NYU
EM-
20
Keller
c.1

Determination of reflected
and transmitted fields...

NYU
EM-
20
c.1

Keller
AUTHOR

Determination of reflected
TITLE
and transmitted fields...

DATE DUE	BORROWER'S NAME	ROOM NUMBER
~~JUN 25 '64~~	~~[illegible]~~	
~~NOV 10 '64~~	~~[illegible]~~	
[illegible]	~~V. K. Park~~	
2·27·87	[illegible]	

N. Y. U. Courant Institute of
Mathematical Sciences
4 Washington Place
New York 3, N. Y.